Finite Source Effects in Microlensing Events

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Abstract

The computation of the magnification of a finite source by an arbitrary gravitational lens can be reduced from a two-dimensional to a one-dimensional integral using a generalization of Stoke's theorem. For a large source lensed by a planetary-system whose planet lies at the position where one of the two images would be in the absence of a planet, the integral can be done analytically. If the planet lies at the position of the major (unperturbed) image, the excess flux is the same as it would be for an isolated planet. If the planet lies at the minor image, there is no excess flux.

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1. Introduction

Four groups have detected more than 100 microlensing events toward the Large Magellanic Cloud and the Galactic bulge (Alcock et al. 1995,1996a; Aubourg et al. 1995; Udalski et al. 1994a; Alard 1996). For most events, the source can be treated as point of light. However, when the source comes sufficiently close to or crosses a caustic (locus of points of infinite magnification in the lens plane), the finite size of the source affects the light curve. One may use these effects to infer the size of the Einstein ring relative to the angular size of the source. Since the latter is generally known from Stefan's law and the color and magnitude of the source, one can then determine the absolute size of the Einstein ring (Gould 1994; Nemiroff & Wickramasinghe 1994). This effect has already been observed for one point-mass lens (Pratt et al. 1996) and for two binary lenses (Udalski et al. 1994b; Alcock et al. 1996b,1996c), and may ultimately be key to measuring the mass function of the lenses (Gould 1996).

For a point-mass lens, one may write the formula for the magnification of a finite source in closed form (Witt & Mao 1994), but for a binary lens, the evaluation is more complicated. In principle, one could compute the magnification at each point of the source and sum these to find the total flux of the images. However, because the magnification is divergent near the caustic, one must take special care in performing the integration in these regions. Since the caustics have a somewhat irregular structure, this form of numerical integration is often difficult.

The problem can be especially acute in the analysis of lensing events by planetary systems because the Einstein ring of a planet is generally of the same order as the size of the source. In order to simulate such events Bennett & Rhie (1996) developed an alternate approach: they examined the points in the image plane (rather than the source plane), calculated the source-plane position for each, and thereby identified all the image points originating in the source. For a source of uniform surface brightness, this method yields the ratio of the total area of the images to the area of the source which, since surface brightness is conserved (Liou-

ville 1837), is equal to the total magnification. The method is easily generalized to non-uniform sources by weighting each point of the image by the local flux of the corresponding point on the source. Here we present a new method for computing the magnification of finite sources.

2. Method

Initially we will assume that the source has uniform surface brightness so that by Liouville's theorem the magnification is just the ratio of the area of the image to the area of the source. Later we will extend the method to more general sources.

Consider first a source that does not cross any caustics. The source will be imaged into m disjoint images. Let C be the boundary of the source and let C'_j be the boundary of the jth image. The parity of each image, $p_j = \pm 1$, is defined as the sign of its magnification tensor. As one moves counter-clockwise around C, one moves counter-clockwise around C'_j for $p_j = 1$ and clockwise for $p_j = -1$. By Stoke's theorem, the area of the source is $(1/2)\int_C \mathbf{r} \times d\mathbf{l}$ and the area of the jth image is $(1/2)p_j \int_{C'_j} \mathbf{r} \times d\mathbf{l}$, where $\mathbf{r}$ is the position on the contour and $d\mathbf{l}$ is the line element. Note that the direction of integration around the image contours is defined by counter-clockwise motion around the source. The magnification is then

$$A = \sum_j p_j \int\limits_{C'_j} \mathbf{r} \times d\mathbf{l} \Big/ \int\limits_{C} \mathbf{r} \times d\mathbf{l}, \tag{2.1}$$

where the two-dimensional cross products are to be regarded as signed scalars.

Equation (2.1) remains valid even when the source crosses one or several caustics. To see this, divide the source into subsources each of which lies entirely inside or entirely outside of caustics. For definiteness, take the case of a binary lens for which the source can be divided into two subsources, one lying inside a caustic and having five images and the other lying outside and having three images. The magnification is then given by the sum of two integrals of the form of equation

(2.1), one integral for each subsource. The difference between this sum and equation (2.1) applied directly to the whole source is eight additional line integrals, five for the image contours mapped from motion in one direction along the inside of the caustic segment, and three for the image contours mapped from motion in the opposite direction along the outside of the caustic segment. Consider first the two images that are present inside but not outside the caustic. These have opposite parities and, for points along the caustic, are mapped into exactly the same points along the critical curve in the image plane. Hence the two line integrals from these images make equal contributions of opposite sign. Now consider the remaining three images. These are unaffected by the presence of the caustic and therefore the contours just inside and just outside the caustic are mapped to the same contours in the image plane. However, since the directions of integration are opposite, the two line integrals cancel for each image. Thus, equation (2.1) is valid for all cases.

3. Application to Planetary Systems

Consider a planet of mass m orbiting a star of mass M, with $m \ll M$. If the planet were not there, the star would lens a background source into two images at positions $\pm x_\pm \theta_e$ where θ_e is the angular Einstein radius of the lensing star,

$$x_\pm \equiv \frac{(x^2+4)^{1/2} \pm x}{2}, \tag{3.1}$$

and $x\theta_e$ is the projected separation between the source and the lens. The magnification tensor is given by

$$\mathcal{M}_\pm = \begin{pmatrix} 1+\gamma_\pm & 0 \\ 0 & 1-\gamma_\pm \end{pmatrix}^{-1}, \qquad \gamma_\pm = x_\mp^2, \tag{3.2}$$

where the (1,1) element represents the magnification along the source-lens axis. The magnification of each image is given by the absolute value of the determinant

of this tensor, $A_\pm = |\mathcal{M}_\pm|$. Note that the shear $\gamma_+ < 1$ for the major image outside the Einstein ring ($x_+ > 1$) and that $\gamma_- = \gamma_+^{-1} > 1$ for the minor image inside the Einstein ring ($x_- < 1$).

We now suppose that the planet lies exactly at the position of one of the two *unperturbed* images of the center of the source. We adopt this position as the center of our coordinates and express all angular distances in units of the Einstein ring of the planet: $\theta_p = (m/M)^{1/2}\theta_e$. We denote positions within the source by $(\rho\cos\psi, \rho\sin\psi)$ and positions within the image by $(r\cos\phi, r\sin\phi)$. We evaluate equation (2.14) from Gould & Loeb (1992), noting that in their notation $(\rho\cos\psi, \rho\sin\psi) = -\epsilon^{-1/2}([1+\gamma]\xi_p, [1-\gamma]\eta_p)$ and $(r\cos\phi, r\sin\phi) = \epsilon^{-1/2}(\xi_i - \xi_p, \eta_i - \eta_p)$. We then find,

$$\rho\cos\psi = \frac{\cos\phi}{r}[r^2(1+\gamma) - 1], \qquad \rho\sin\psi = \frac{\sin\phi}{r}[r^2(1-\gamma) - 1]. \tag{3.3}$$

Squaring and adding these two equations yields a quadratic equation in r^2, the two solutions of which are

$$r_\pm^2 = \frac{b \pm (b^2 - 4a)^{1/2}}{2a}, \quad a \equiv 1 + \gamma^2 + 2\gamma\cos 2\phi, \quad b \equiv \rho^2 + 2 + 2\gamma\cos 2\phi. \tag{3.4}$$

Suppose that the source is large enough so that it covers all caustics (see e.g. fig. 3 from Gould & Loeb 1992). The boundary of the source will then always have two images, one at r_+ and one at r_-. Using equation (2.1), and assuming that the source has constant surface brightness, we find a magnification

$$A = \frac{1}{2\pi\rho^2}\int_0^{2\pi} d\phi\,(r_+^2 - r_-^2) = \frac{1}{|1-\gamma^2|} + \frac{1 + \mathrm{sgn}(1-\gamma)}{\rho^2} - \frac{q}{\rho^4} + ... \tag{3.5}$$

where $\mathrm{sgn}(1-\gamma_\pm) = \pm 1$ and $q = [(1/2 + \rho^{-2})^2 - \gamma^2\rho^{-4}]^{-1/2}$. The first term is just the magnification of the source in the absence of a planet [cf. eq. (3.2)]. For the major image, the second term is $2\rho^{-2}$. Thus, for a source of unit surface brightness

the total excess flux is $2\pi\theta_p^2$, exactly the same as the result for an isolated planet. On the other hand, to this order there is no excess flux when a planet perturbs the minor image, a result already suggested by the numerical calculations of Bennett & Rhie (1996). Successive additional terms are each smaller by ρ^{-2}.

4. Numerical Integration

To translate equation (2.1) into a prescription for numerical integration, we first approximate the boundary of the source as a polygon of n (not necessarily equal) sides. We denote the (two-component) vertices in counter-clockwise order by $\mathbf{s}_0$, $\mathbf{s}_1$ ... $\mathbf{s}_n$, with $\mathbf{s}_n = \mathbf{s}_0$. For each source vertex $\mathbf{s}_i$, there will be a variable number of image positions $\mathbf{u}_{i,j}$. The vertex images should be ordered so that $\mathbf{u}_{i-1,j}$ and $\mathbf{u}_{i,j}$ lie on the same image curve. When the source contour crosses a caustic and two images disappear, these images should be replaced by "blanks". When a caustic is crossed and two new images appear, they should be entered into previously blank columns. With this ordering, the parities of the image vertices depend only on j: $p_{i,j} \rightarrow p_j$. For simplicity, we initially assume that if any caustics are crossed, one of the source vertices is chosen to lie right on the caustic.

The magnification is then given by,

$$A = \sum_{i=1}^{n} \sum_{j'} p_j (\mathbf{u}_{i-1,j} \times \mathbf{u}_{i,j}) \Big/ \sum_{i=1}^{n} \mathbf{s}_{i-1} \times \mathbf{s}_i, \tag{4.1}$$

where the prime in j' indicates that there is no summation for first appearance of new images at a caustic (in which case there is, of course, no previous image position $\mathbf{u}_{i-1,j}$).

In equation (4.1), we assumed that if the source boundary crossed a caustic (moving counter-clockwise) thereby created or destroying two images, then one of the vertices would be chosen to lie exactly on the caustic. We now show that if the first point does not lie on the caustic, there is a simple prescription which in

effect replaces the two terms connecting the critical curve and the two images of the first point inside the caustic with a single term that connects the two image points directly. Let j and $j+1$ be two new images and let $\mathbf{s}_i$ be chosen to lie exactly on the caustic where they are created. The first term to be included in the sum for the j image would be $i+1$ and this term would include the boundary between the critical curve (at $\mathbf{u}_{i,j}$) and the point at $\mathbf{u}_{i+1,j}$. The situation is similar for image $j+1$. The two new images have opposite parities, $p_{j+1} = -p_j$. Because $\mathbf{s}_i$ lies on the caustic, $\mathbf{u}_{i,j} = \mathbf{u}_{i,j+1}$. The sum of the first terms for these two new images will then be

$$p_j(\mathbf{u}_{i,j} \times \mathbf{u}_{i+1,j}) + p_{j+1}(\mathbf{u}_{i,j+1} \times \mathbf{u}_{i+1,j+1}) = p_j \mathbf{u}_{i,j} \times (\mathbf{u}_{i+1,j} - \mathbf{u}_{i+1,j+1}). \quad (4.2)$$

To a good approximation $\mathbf{u}_{i,j} = (\mathbf{u}_{i+1,j} + \mathbf{u}_{i+1,j+1})/2$, so one may simply replace the two terms on the left-hand side of equation (4.2) with $p_j \mathbf{u}_{i+1,j+1} \times \mathbf{u}_{i+1,j}$. Now let $\mathbf{s}_{i'}$ be the vertex on a caustic where the two images disappear. Using a similar argument, one can show that the two last terms for these images can be replaced by $-p_j \mathbf{u}_{i'-1,j+1} \times \mathbf{u}_{i'-1,j}$. Hence, it is not actually necessary to have vertices on the caustics. Suppose that there are k caustic crossings, $\ell = 1...k$ where two images j_ℓ and $j_\ell + 1$ are created, and k other crossings where they are destroyed. Let the first point after the images have been created be i_ℓ and last before they are destroyed be i'_ℓ. If these first and last points do not lie on the caustic, then the numerator in equation (4.1) should be replaced by

$$\rightarrow \sum_{i=1}^{n} \sum_{j'} p_j(\mathbf{u}_{i-1,j} \times \mathbf{u}_{i,j}) + \sum_{\ell=1}^{k} p_{j_\ell}[\mathbf{u}_{i_\ell,j_\ell+1} \times \mathbf{u}_{i_\ell,j_\ell} - \mathbf{u}_{i'_\ell,j_\ell+1} \times \mathbf{u}_{i'_\ell,j_\ell}]. \quad (4.3)$$

5. Discussion

Some of the most interesting applications of finite source effects in microlensing involve the color changes due to differential limb darkening (Witt 1995). For example, this effect can be exploited to measure the Einstein ring size even when single-band photometric effects are undetectable, and it is especially useful in understanding planetary events (Loeb & Sasselov 1995; Gould & Welch 1996). The method given above cannot be directly applied to limb darkened stars since constant surface brightness was assumed. However, one could model the source star as being composed of rings of constant surface brightness, and each ring could be evaluated by taking the difference of fluxes due to sources contained within two successive rings.

The method given here is simpler than that of Bennett & Rhie (1996) in that it requires only a one-dimensional integral, but it is more complicated in that one must find the individual image positions corresponding to the source boundary. (One must also find the parity and hence the magnification, but this need be done only once for each image contour.) The method of choice therefore depends on the lens system. For planetary system lenses, it is often possible to treat the effect of the planet as a perturbation on the background shear generated by its parent star. In these cases, the lens equation can be reduced to a quartic equation (Gould & Loeb 1992) which can be solved analytically. The method given here is therefore far more efficient. In general, binary lenses require solution of a fifth order equation. There are standard packages that do this, but they require substantially longer computations than does the quartic case. Nevertheless, given that one is searching for images of points along the continuous one dimensional boundary of the source, it should be possible to speed up the fifth order programs by using the solution found for one point as a trial solution for the next. However, for more complicated lenses, two-dimensional integration over the image plane may be preferable.

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